Lagrange's Theorem

Sasha Patotski

Cornell University

ap744@cornell.edu

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Group action definition

Definition

Let G be a group and X be a set. An **action** of G on X is a homomorphism $G \to Bij(X)$.

- Equivalently, **action** of G on X is a map $G \times X \to X$, $(g, x) \mapsto g.x$ such that $g.(h.x) = (gh).x$ and $e.x = x$.

Definition

Let G be a group acting on a set X. The **orbit** of $x \in X$ is the set $Gx \subseteq X$, i.e. $Gx = \{g.x \mid g \in G\}$.
For a point $x \in X$, its **stabilizer** G_x is the set $G_x = \{g \in G \mid g.x = x\}$.

Abstract groups are not that abstract

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Any abstract group G is actually a transformation group.

Indeed, take $X = G$ with $G \times X \to X$ being the multiplication map.

In other words, to a $g \in G$ we associate a function $L_g \colon X \to X$, mapping $h \in X = G$ to $L_g(h) := gh$.

This defines **injective** homomorphism $\varphi \colon G \to Bij(X)$, $g \mapsto L_g$.

Corollary: any finite group is a subgroup of S_n for some n.

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- In other words, a **left** coset of H in G is gH where $g \in G$ (H is on the **right**).

  A **right** coset of H in G is Hg where $g \in G$ (H is on the **left**).

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- Let $G = S_3$ and $H = \{1, (12)\}$. What are the cosets?

Theorem

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Proof: $|G|/|H|$ is the number of left (or right) cosets, and so is an integer.

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This subgroup has $m!(n-m)!$ elements, and Lagrange's theorem gives the result.

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There is a subgroup of S_{ab} of permutations allowed to permute the above groups of numbers, and after that only to permute numbers within each group. Note that it's **not!** just $S_b \times S_a^b$.

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There is a subgroup of S_{ab} of permutations allowed to permute the above groups of numbers, and after that only to permute numbers within each group. Note that it's **not!** just $S_b \times S_a^b$.
It has $b!(a!)^b$ elements, and Lagrange Theorem gives the proof of the second statement.

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This subgroup has size 2, so by Lagrange's theorem the number of elements in $(\mathbb{Z}/m)^\times$ is even.

Polya's Enumeration Theorem

Theorem

Suppose that a finite group G acts on a finite set X. Then the number of colorings of X in n colors inequivalent under the action of G is

$$N(n) = \frac{1}{|G|} \sum_{g \in G} n^{c(g)}$$

where $c(g)$ is the number of cycles of g as a permutation of X.

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- There are 2 reflections with 2 cycles, and 2 reflections with 3 cycles, with contribute $2 \cdot 2^2 + 2 \cdot 2^3 = 24$.

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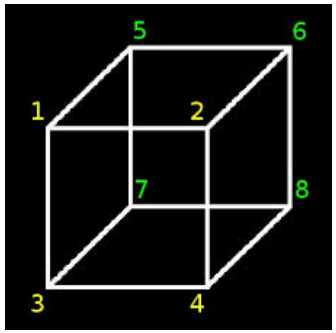
- Summing up, $N(2) = \frac{1}{8}(16 + 4 + 4 + 24) = 6$.
- For n colors, $N(n) = \frac{n^4+2n^3+3n^2+2n}{8}$. For example, $N(4) = 55$.

One more example



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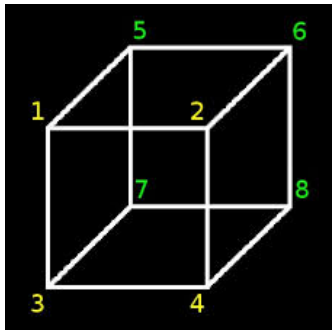

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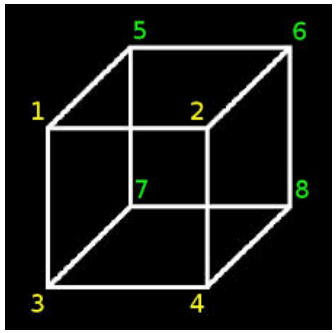

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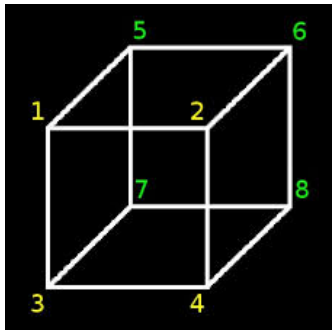

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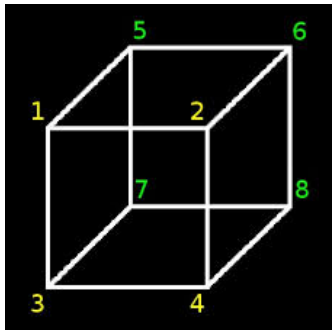

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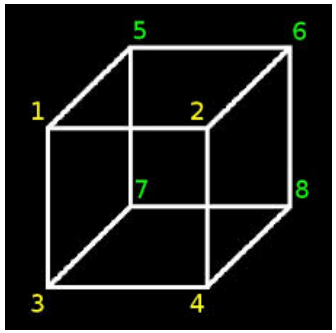

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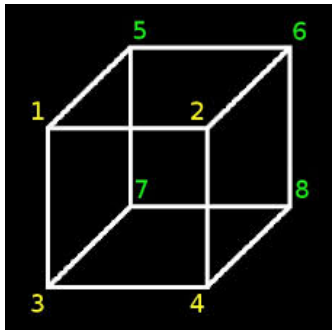

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